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Levitin–Polyak well-posedness of constrained vector optimization problems

X. X. Huang · X. Q. Yang

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Abstract In this paper, we consider Levitin–Polyak type well-posedness for a general constrained vector optimization problem. We introduce several types of (generalized) Levitin–Polyak well-posednesses. Criteria and characterizations for these types of well-posednesses are given. Relations among these types of well-posedness are investigated. Finally, we consider convergence of a class of penalty methods under the assumption of a type of generalized Levitin–Polyak well-posedness.

Keywords Constrained vector optimization · Minimizing sequence · (generalized) Levitin–Polyak well-posedness · Penalty type methods

1 Introduction

The study of well-posedness started from Tykhonov [16] and Levitin and Polyak [12]. Since then, various notions of well-posedness have been defined and extensively studied (see, e.g. [5, 10, 17] and the references therein). It is worth noting that recent studies on well-posedness have been extended to vector optimization problems (see, e.g. [3, 7, 14] and the references therein). The study of Levitin–Polyak well-posedness for convex scalar optimization problems with explicit constraints originates from [10]. Most recently, this research was extended to nonconvex optimization problems with explicit constraints (Huang and Yang, Submitted).

Let (X, d_1) and (Z, d_2) be two metric spaces. Let Y be a normed space ordered by a closed and convex cone C with nonempty interior int$C$, i.e., $\forall y_1, y_2 \in Y$, $y_1 \leq_C y_2$ if and only if $y_2 - y_1 \in C$. Arbitrarily fix an $e \in \text{int}C$. Let $X_1 \subset X$ and $K \subset Z$ be two

X. X. Huang (✉)
School of Management, Fudan University, Shanghai 200433, China
e-mail: xxhuang@fudan.edu.cn

X. Q. Yang
Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong

nonempty and closed sets. Consider the following constrained vector optimization problem:

$$\text{(VP)} \quad \inf f(x) \quad \text{s.t. } x \in X_1,\ g(x) \in K,$$

where $f: X \to Y$ and $g: X \to Z$ are continuous functions.

Denote by X_0 the set of feasible solutions of (VP), i.e.,

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Throughout the paper, we always assume that $X_0 \neq \emptyset$.

Denote by X^* the set of weakly efficient solutions of (VP), namely, for any $x^* \in X^*$,

(1) $x^* \in X_0$ and
(2) for any $x \in X_0, f(x) - f(x^*) \notin -\text{int}C$.

We denote by V the set of infimal points of (VP). That is, $v \in V$ if and only if

(1) there exists no $x \in X_0$ such that $f(x) - v \in \text{int}C$;
(2) there exists a sequence $\{x_k\} \subset X_0$ such that $f(x_k) \to v$.

Throughout the paper, we always assume that $V \neq \emptyset$. Let (P,d) be a metric space and $P_1 \subset P$. We denote by $d_{P_1}(p) = \inf\{d(p,p') : p' \in P_1\}$ the distance from the point p to the set P_1.

Define

$$\xi(y) = \min\{t : y \leq_C te\}, \quad \forall y \in Y.$$

It is known from [13] that ξ is continuous, homogenous, (strictly) monotone (i.e., $\xi(y_1) \leq \xi(y_2)$ if $y_2 - y_1 \in C$) and $\xi(y_1) < \xi(y_2)$ if $y_2 - y_1 \in \text{int}C$) and convex.

Many optimization methods for (VP) may generate a sequence $\{x_k\} \subset X_1$ such that $d_{X_0}(x_k) \to 0$.

Penalty type methods for (VP) (and its special cases, e.g. $Y = R^l, C = R^l_+$), such as penalty function methods (see, e.g. [9]) and augmented Lagrangian methods (see, e.g. [8]) may generate a sequence $\{x_k\} \subset X_1$ such that $d_K(g(x_k)) \to 0$, but $d_{X_0}(x_k) \not\to 0$.

In this paper, we will study such sequences under additional conditions. This study should be useful to the study of convergence of some optimization methods for (VP) as will be seen in Sect. 4 of this paper.

In what follows, we will introduce several notions of Levitin–Polyak well-posedness and generalized Levitin–Polyak wells-posedness for (VP).

Definition 1.1

(1) (VP) is said to be type I Levitin–Polyak (LP in short) well-posed if $X^* \neq \emptyset$ and, for any $\{x_k\}$ satisfying

$$d_{X_0}(x_k) \to 0 \tag{1}$$

and

$$d_V(f(x_k)) \to 0, \tag{2}$$

there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in X^*$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

(2) (VP) is said to be type I LP well-posed in the generalized sense if $X^* \neq \emptyset$ and, for any $\{x_k\}$ satisfying

$$d_K(g(x_k)) \to 0 \tag{3}$$

and (2),
there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in X^*$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

The sequence satisfying (1) and (2) is called a type I LP minimizing one while the sequence satisfying (3) and (2) is called a type I generalized LP minimizing one.

Definition 1.2

(1) (VP) is said to be type II LP well-posed if $X^* \neq \emptyset$ and, for any $\{x_k\}$ satisfying (1) and

$$f(x_k) \leq_C v_k + \epsilon_k e \quad \text{for some } \{v_k\} \subset V \quad \text{and some } 0 < \epsilon_k \to 0, \tag{4}$$

there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in X^*$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

(2) (VP) is said to be type II LP well-posed in the generalized sense if $X^* \neq \emptyset$ and, for any $\{x_k\}$ meeting (3) and (4), then there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in X^*$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

The sequence satisfying (1) and (4) is called a type II LP minimizing one while the sequence satisfying (3) and (4) is called a type II generalized LP minimizing one.

Definition 1.3

(1) (VP) is said to be type III LP well-posed if $X^* \neq \emptyset$ and, for any $\{x_k\}$ satisfying (1) and

$$\liminf_{k \to +\infty} \left\{ \inf_{v \in V} \xi(v - f(x_k)) \right\} \geq 0, \tag{5}$$

there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in X^*$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

(2) (VP) is said to be type III LP well-posed in the generalized sense if $X^* \neq \emptyset$ and for any $\{x_k\}$ meeting (3) and (5), then there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in X^*$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

The sequence satisfying (1) and (5) is called a type III LP minimizing one while the sequence satisfying (3) and (5) is called a generalized LP minimizing one.

Remark 1

(1) The definitions of types I (condition (2)), II (condition (4)) and III (condition (5)) (generalized) LP minimizing sequence were motivated by Definitions 2.3–2.5 of [6].
(2) It is easy to see that a type I (generalized) LP minimizing sequence is a type II generalized LP minimizing sequence and that a type II (generalized) LP minimizing sequence is a type III (generalized) LP minimizing sequence. Thus, the type III (generalized) LP well-posedness implies the type II (generalized) LP well-posedness and the type II (generalized) LP well-posedness implies the type I (generalized) LP well-posedness.
(3) Any type of (generalized) well-posedness implies that the set X^* of weakly efficient solutions of (VP) is nonempty and compact.
(4) When $Y = R^1$, $C = R^1_+$, type I (generalized) LP well-posedness coincides with type II (generalized) LP well-posedness, type I (II) LP well-posedness is just the LP well-posedness in (Huang and Yang, submitted) while type I (II) generalized LP well-posedness is the generalized LP well-posedness defined in (Huang and Yang, submitted), and type III generalized LP well-posedness is just the strongly generalized LP well-posedness in (Huang and Yang, submitted).

The paper is organized as follows. In Sect. 2, we present some criteria and characterizations for the various (generalized) LP well-posednesses. Section 3 gives the relations among these types of (generalized) LP well-posednesess. Section 4 presents an application of a generalized LP well-posedness to the convergence of a class of penalty methods for (VP).

2 Criteria and characterizations for (generalized) Lp well-posedness

In this section, we give necessary and sufficient conditions for the various types of (generalized) LP well-posedness defined in Sect. 1.

Consider the following statement:

$$\begin{aligned}&[X^* \neq \emptyset \text{ and, for any type I (resp. types II, and III,}\\&\text{generalized types I–III)}\\&\text{ LP minimizing sequence } \{x_k\}, \text{ we have } d_{X^*}(x_k) \to 0].\end{aligned} \tag{6}$$

First, we have the following result, whose proof is elementary and thus omitted.

Proposition 2.1 *If* (VP) *is type I (resp. types II and III, generalized types I–III) LP well-posed, then* (6) *holds. Conversely, if* (6) *holds and* X^* *is compact, then* (VP) *is type I (resp. types* II*, and* III*, generalized types I–III) LP well-posed.*

Now consider a real-valued function $c = c(t,s)$ defined for $t,s \geq 0$ sufficiently small, such that

$$c(t,s) \geq 0, \quad \forall t,s, \quad c(0,0) = 0, \tag{7}$$

$$s_k \to 0, t_k \geq 0, c(t_k,s_k) \to 0 \text{ imply } t_k \to 0. \tag{8}$$

Theorem 2.1 *If* (VP) *is type* I *LP well-posed, then there exists a function c satisfying* (7) *and* (8) *such that*

$$d_V(f(x)) \geq c(d_{X^*}(x), d_{X_0}(x)), \quad \forall x \in X_1. \tag{9}$$

Conversely, suppose that X^* is nonempty and compact, and (9) holds for some c satisfying (7) and (8). Then (VP) is type I LP well-posed.

Proof Define

$$c(t,s) = \inf\{d_V(f(x)) : x \in X_1, d_{X^*}(x) = t, d_{X_0}(x) = s\}.$$

It is obvious that $c(t,s) \geq 0, \forall s,t$ and $c(0,0) = 0$. Moreover, if $s_k \to 0$, $t_k \geq 0$ and $c(t_k, s_k) \to 0$, then, there exists a sequence $\{x_k\} \subset X_1$ with

$$d_{X^*}(x_k) = t_k, \tag{10}$$

$$d_{X_0}(x_k) = s_k, \tag{11}$$

such that

$$d_V(f(x_k)) \to 0. \tag{12}$$

Note that $s_k \to 0$. This fact together with (11) and (12) implies that $\{x_k\}$ is a type I LP minimizing sequence. By Proposition 2.1, we have $t_k \to 0$. This completes the proof of the first part of the theorem. Conversely, let $\{x_k\}$ be a type I LP minimizing sequence. Then, by (9), we have

$$d_V(f(x_k)) \geq c(d_{X^*}(x_k), d_{X_0}(x_k)), \quad \forall k. \tag{13}$$

Let

$$t_k = d_{X^*}(x_k), \quad s_k = d_{X_0}(x_k).$$

Then, $s_k \to 0$. In addition, $d_V(f(x_k)) \to 0$. These facts together with (13) as well as the properties of the function c imply that $t_k \to 0$. By Proposition 2.1, we see that (VP) is type I LP well-posed. □

Theorem 2.2 *If* (VP) *is type* I *LP well-posed in the generalized sense, then there exists a function c satisfying* (7) *and* (8) *such that*

$$d_V(f(x)) \geq c(d_{X^*}(x), d_K(g(x))), \quad \forall x \in X_1. \tag{14}$$

Conversely, suppose that X^* is nonempty and compact, and (14) holds for some c satisfying (7) and (8). Then (VP) is type I LP well-posed in the generalized sense.

Proof The proof is almost the same as that of Theorem 2.1. The only difference lies in the proof of the first part of Theorem 2.1. Here, we define

$$c(t,s) = \inf\{d_V(f(x)) : x \in X_1, d_{X^*}(x) = t, d_K(g(x)) = s\}.$$ □

Furi and Vignoli [6] characterized well-posedness of optimization problems (defined in a complete metric space (X, d_1)) by making use of the Kuratowski measure of noncompactness of a subset A of X defined by

$$\alpha(A) = \inf\left\{\epsilon > 0 : A \subset \bigcup_{1 \leq i \leq n} C_i, \text{ for some } C_i, \operatorname{diam}(C_i) \leq \epsilon\right\},$$

where $\mathrm{diam}(C_i)$ is the diameter of C_i defined by

$$\mathrm{diam}(C_i) = \sup\{d_1(x_1, x_2) : x_1, x_2 \in C_i\}.$$

Given two nonempty subsets A and B of X, define the excess of set A to set B by

$$e(A, B) = \sup\{d_B(a) : a \in A\}.$$

The Hausdorff distance between A and B is defined as

$$\mathrm{haus}(A, B) = \max\{e(A, B), e(B, A)\}.$$

Next we give Furi–Vignoli type characterizations for the various (generalized) LP well-posednesses.

Let, for each $\epsilon > 0$,

$$T_1^1(\epsilon) = \{x \in X_1 : d_V(f(x)) \leq \epsilon, d_{X_0}(x) \leq \epsilon\}.$$

Theorem 2.3 *Let* (X, d_1) *be a complete metric space and* $V \neq \emptyset$. *Then* (VP) *is type* I *LP well-posed if and only if*

$$\lim_{\epsilon \to 0} \alpha(T_1^1(\epsilon)) = 0. \tag{15}$$

Proof First, we show that for each $\epsilon > 0$, $T_1^1(\epsilon)$ is nonempty and closed. The nonemptiness of $T_1^1(\epsilon)$ follows from the fact that $V \neq \emptyset$. Let $\{x_k\} \subset T_1^1(\epsilon)$ and $x_k \to \bar{x}$. Then

$$d_V(f(x_k)) \leq \epsilon \tag{16}$$

and

$$d_{X_0}(x_k) \leq \epsilon. \tag{17}$$

From (17), we have

$$d_{X_0}(\bar{x}) \leq \epsilon. \tag{18}$$

By the continuity of f and (16), we obtain

$$d_V(f(\bar{x})) \leq \epsilon. \tag{19}$$

The combination of (18) and (19) shows that $\bar{x} \in T_1^1(\epsilon)$. Thus, $T_1^1(\epsilon)$ is closed.

Second, we show that

$$X^* = \cap_{\epsilon > 0} T_1^1(\epsilon). \tag{20}$$

It is obvious that $X^* \subset \cap_{\epsilon>0} T_1^1(\epsilon)$. Now suppose that $\epsilon_k \to 0$ and $x^* \in \cap_{k=1}^{\infty} T_1^1(\epsilon_k)$. Then,

$$d_V(f(x^*)) \leq \epsilon_k, \quad \forall k \tag{21}$$

and

$$d_{X_0}(x^*) \leq \epsilon_k, \quad \forall k. \tag{22}$$

By (21), we have $f(x^*) \in V$. By (22), we have $x^* \in X_0$. Hence, $x^* \in X^*$.

Now we assume that (15) holds. Clearly, $T_1^1(\cdot)$ is increasing with $\epsilon > 0$. By the Kuratowski theorem ([11], p 318), we have

$$\text{haus}(T_1^1(\epsilon), T_1^1) \to 0 \quad \text{as } \epsilon \to 0, \tag{23}$$

where

$$T_1^1 = \cap_{\epsilon>0} T_1^1(\epsilon)$$

is nonempty and compact.

Let $\{x_k\}$ be a type I LP minimizing sequence. Then, by taking a subsequence, we can find a decreasing sequence $\epsilon_k \to 0$ such that $d_V(f(x_k)) \leq \epsilon_k$ and $d_{X_0}(x_k) \leq \epsilon_k$. Thus, $x_k \in T_1^1(\epsilon_k)$. It follows from (20) and (23) that $d_{X^*}(x_k) \to 0$. By Proposition 2.1 (VP) is type I LP well-posed.

Conversely, let (VP) be type I LP well-posed. Consider the excess

$$q(\epsilon) = e(T_1^1(\epsilon), X^*), \quad \epsilon > 0.$$

We show that $q(\epsilon) \to 0$ as $\epsilon \to 0$. If not, there exist $\delta > 0$, $\epsilon_k \to 0$, $x_k \in T_1^1(\epsilon_k)$ such that

$$d_{X^*}(x_k) \geq \delta, \quad \forall k,$$

contradicting the type I LP well-posedness of (VP). Thus, $q(\epsilon) \to 0$ as $\epsilon \to 0$. Note that

$$T_1^1(\epsilon) \subset \{x \in X_1 : d_{X^*}(x) \leq q(\epsilon)\}.$$

It follows that

$$\alpha(T_1^1(\epsilon)) \leq 2q(\epsilon).$$

since $\alpha(X^*) = 0$. Consequently (15) holds. The proof is complete. □

Consider

$$T_1^2(\epsilon) = \{x \in X_1 : d_V(f(x)) \leq \epsilon, d_K(g(x)) \leq \epsilon\}.$$

The following theorem can be proved analogously to Theorem 2.3.

Theorem 2.4 *Let (X, d_1) be a complete metric space and $V \neq \emptyset$. Then* (VP) *is type* I *LP well-posed in the generalized sense if and only if*

$$\lim_{\epsilon\to 0} \alpha(T_1^2(\epsilon)) = 0. \tag{24}$$

Define

$$T_2^1(\epsilon) = \{x \in X_1 : d_{X_0}(x) \leq \epsilon, f(x) \leq_C v + \epsilon e \text{ for some } v \in V\}.$$

Theorem 2.5 *Let (X, d_1) be a complete metric space and $V \neq \emptyset$. Then* (VP) *is type* II *LP well-posed if and only if*

$$\lim_{\epsilon\to 0} \alpha(T_2^1(\epsilon)) = 0. \tag{25}$$

Proof It is obvious from $V \neq \emptyset$ that $T_2^1(\epsilon) \neq \emptyset, \forall \epsilon > 0$. Thus, $clT_2^1(\epsilon)$ is nonempty and closed. Of course, $clT_2^1(\cdot)$ is increasing with ϵ. Now we show that

$$X^* = \cap_{\epsilon>0} clT_2^1(\epsilon). \tag{26}$$

Obviously, $X^* \subset \cap_{\epsilon>0} clT_2^1(\epsilon)$. Let $x^* \in \cap_{\epsilon>0} clT_2^1(\epsilon)$ and $\epsilon_k \downarrow 0$. By $x^* \in \cap_{k=1}^{\infty} clT_2^1(\epsilon_k)$, for each k, there exist $x_{k,j} \in X_1$ and $v_{k,j} \in V$ such that

$$f(x_{k,j}) \leq_C v_{k,j} + \epsilon_k e, \tag{27}$$

$$x_{k,j} \to x^* \tag{28}$$

and

$$d_{X_0}(x_{k,j}) \leq \epsilon_k \Rightarrow d_{X_0}(x^*) \leq \epsilon_k. \tag{29}$$

From (27) and (28) and the continuity of f, we have that for each k, there exists $j(k)$ such that

$$f(x^*) \leq_C v_{k,j(k)} + 2\epsilon_k e. \tag{30}$$

Suppose to the contrary that $x^* \notin X^*$. Then there exist $x_0 \in X_0$ and $\delta > 0$ such that

$$f(x_0) \leq_C f(x^*) - \delta e. \tag{31}$$

From (30) and (31), we have

$$\begin{aligned} f(x_0) &\leq_C v_{k,j(k)} + 2\epsilon_k e - \delta e \\ &= v_{k,j(k)} - (\delta - 2\epsilon_k)e. \end{aligned} \tag{32}$$

Since $\epsilon_k \downarrow 0$, $\delta - 2\epsilon_k \geq \delta/2$ when k is sufficiently large. Thus (32) contradicts the fact that $v_{k,k(j)} \in V$ when k is sufficiently large. Hence, there holds $x^* \in X^*$. Thus (26) is proved.

Now assume that (24) holds. Then

$$\alpha(clT_2^1(\epsilon)) = \alpha(T_2^1(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0.$$

By the Kuratowski theorem, it follows that

$$\text{haus}(clT_2^1(\epsilon), T_2^1) \to 0 \quad \text{as } \epsilon \to 0, \tag{33}$$

where

$$T_2^1 = \cap_{\epsilon>0} clT_2^1(\epsilon)$$

is nonempty and compact. Let $\{x_k\}$ be a type II LP minimizing sequence. Then, by taking a subsequence, we can find a decreasing sequence $\epsilon_k \to 0$ and a sequence $\{v_k\} \subset V$ such that

$$f(x_k) \leq_C v_k + \epsilon_k e, \tag{34}$$
$$d_{X_0}(x_k) \leq \epsilon_k. \tag{35}$$

From (34) and (35), we see that $x_k \in T_2^1(\epsilon_k)$. It follows from (26) and (33) that $d_{X^*}(x_k) \to 0$. By Proposition 2.1 and the compactness of X^*, we deduce that (VP) is type II LP well-posed. The proof of the second part of the theorem is similar to that of the second part of Theorem 2.3. □

Let

$$T_2^2(\epsilon) = \{x \in X_1 : d_K(g(x)) \leq \epsilon, f(x) \leq_C v + \epsilon e \text{ for some } v \in V\}.$$

The next theorem can be proved analogously to Theorem 2.5.

Theorem 2.6 *Let* (X, d_1) *be a complete metric space and* $V \neq \emptyset$. *Then* (VP) *is type II LP well-posed in the generalized sense if and only if*

$$\lim_{\epsilon \to 0} \alpha(T_2^2(\epsilon)) = 0.$$

Definition 2.4 (VP) is said to be inf-externally stable if for each $x_0 \in X_0$, there exists $v_0 \in V$ such that $v_0 \leq_C f(x_0)$.

Define

$$T_3^1(\epsilon) = \{x \in X_1 : \inf_{v \in V} \xi(v - f(x)) \geq -\epsilon, d_{X_0}(x) \leq \epsilon\}.$$

Theorem 2.7 *Let* (X, d_1) *be a complete metric space and* $V \neq \emptyset$. *Suppose that (VP) is inf-externally stable. Then (VP) is type III LP well-posed if and only if*

$$\lim_{\epsilon \to 0} \alpha(T_3^1(\epsilon)) = 0.$$

Proof First, we show that $T_3^1(\epsilon)$ is nonempty and closed for any $\epsilon > 0$. The nonemptiness of $T_3^1(\epsilon)$ follows from the fact that $V \neq \emptyset$. Now let $\{x_k\} \subset T_3^1(\epsilon)$ and $x_k \to \bar{x}$. Then,

$$\inf_{v \in V} \xi(v - f(x_k)) \geq -\epsilon, \tag{36}$$

$$d_{X_0}(x_k) \leq \epsilon. \tag{37}$$

Note that the continuity of f implies that the function $\inf_{v \in V} \xi(v - f(\cdot))$ is upper semicontinuous. Taking the upper limit in (36), we have

$$\inf_{v \in V} \xi(v - f(\bar{x})) \geq -\epsilon. \tag{38}$$

Taking the limit in (37), we obtain

$$d_{X_0}(\bar{x}) \leq \epsilon. \tag{39}$$

The combination of (38) and (39) yields $\bar{x} \in T_3^1(\epsilon)$. Hence, $T_3^1(\epsilon)$ is closed.

Second, we show that

$$X^* = \cap_{\epsilon > 0} T_3^1(\epsilon). \tag{40}$$

Obviously, $X^* \subset \cap_{\epsilon > 0} T_3^1(\epsilon)$. Now let $x^* \in \cap_{\epsilon > 0} T_3^1(\epsilon)$ and $\epsilon_k \downarrow 0$. Then

$$\inf_{v \in V} \xi(v - f(x^*)) \geq -\epsilon_k, \tag{41}$$

$$d_{X_0}(x^*) \leq \epsilon_k. \tag{42}$$

From (42), we have $x^* \in X_0$. From (41), we have

$$\xi(v - f(x^*)) \geq 0, \quad \forall v \in V. \tag{43}$$

Suppose to the contrary that there exist $x_0 \in X_0$ and $\delta > 0$ such that

$$f(x_0) - f(x^*) \leq_C -\delta e. \tag{44}$$

By the inf-external stability of (VP), there exists $v_0 \in V$ such that $v_0 \leq_C f(x_0)$. This together with (44) implies that

$$\xi(v_0 - f(x^*)) \leq -\delta,$$

contradicting (43). Thus (40) is proved. Clearly, $T_3^1(\cdot)$ is increasing with $\epsilon > 0$. By the Kuratowski theorem, we have

$$\text{haus}(T_3^1(\epsilon), T_3^1) \to 0 \quad \text{as} \epsilon \to 0, \tag{45}$$

where

$$T_3^1 = \cap_{\epsilon > 0} T_3^1(\epsilon)$$

is nonempty and compact.

Let $\{x_k\}$ be a type III LP minimizing sequence. Then, by taking a subsequence, we can find a decreasing sequence $\epsilon_k \to 0$ such that

$$\inf_{v \in V} \xi(v - f(x_k)) \geq -\epsilon_k,$$

$$d_{X_0}(x_k) \leq \epsilon_k.$$

Thus, $x_k \in T_3^1(\epsilon_k)$. By (40) and (45) we see that $d_{X^*}(x_k) \to 0$. By Proposition 2.1 (VP) is type III LP well-posed. The second part of the theorem can be proved similarly to that of Theorem 2.3. The proof is complete. □

Define

$$T_3^2(\epsilon) = \left\{ x \in X_1 : \inf_{v \in V} \xi(v - f(x)) \geq -\epsilon, d_K(g(x)) \leq \epsilon \right\}.$$

The following theorem can be proved analogously to Theorem 2.7.

Theorem 2.8 *Let (X, d_1) be a complete metric space and $V \neq \emptyset$. Suppose that (VP) is inf-externally stable. Then (VP) is type III LP well-posed in the generalized sense if and only if*

$$\lim_{\epsilon \to 0} \alpha(T_3^2(\epsilon)) = 0.$$

Next proposition gives sufficient conditions for the type III (generalized) LP well-posedness.

Proposition 2.2

(1) *Assume that there exists $\delta > 0$ such that*

$$X_1(\delta) = \{x \in X_1 : d_{X_0}(x) \leq \delta\} \tag{46}$$

is compact. Then, (VP) *is type* III *LP well-posed.*

(2) *Assume that there exists* $\delta > 0$ *such that*

$$X_2(\delta) = \{x \in X_1 : d_K(g(x)) \leq \delta\} \tag{47}$$

is compact. Then (VP) is type III LP well-posed in the generalized sense.

Proof We prove only (1) and (2) can be similarly proved.

Let $\{x_k\}$ be a type III LP minimizing sequence. Then

$$\liminf_{k\to+\infty}\{\inf_{v\in V} \xi(v - f(x_k))\} \geq 0, \tag{48}$$

$$d_{X_0}(x_k) \to 0. \tag{49}$$

(49) implies that $x_k \in X_1(\delta)$ when $k \geq k_0$ for some $k_0 > 0$. By the compactness of $X_1(\delta)$, there exist a subsequence $\{x_{k_j}\}$ and $x^* \in X_1(\delta)$ such that $x_{k_j} \to x^*$. This together with (49) implies that $x^* \in X_0$. Moreover, from (48), we have

$$\xi(v - f(x^*)) \geq 0, \quad \forall v \in V. \tag{50}$$

Suppose to the contrary that $x^* \notin X^*$. Then, there exists $x_0 \in X_0$ such that

$$f(x_0) - f(x^*) \in -\text{int}C. \tag{51}$$

Note that $X_0 \subset X_1(\delta)$ is nonempty and compact and f is continuous. Consequently, there exists $v_0 \in V$ such that

$$v_0 \leq_C f(x_0). \tag{52}$$

The combination of (50)–(52) leads to a contradiction. Hence, $x^* \in X^*$ and the proof is complete. □

Now we consider the special case when X is a finite dimensional normed space, $Y = R^l$, $C = R^l_+$, $e = (1, \ldots, 1) \in R^l$, $\xi(y) = \max\{y_i : i = 1, \ldots, l\}, \forall y \in Y$.

Definition 2.2 Let X be a finite dimensional normed space, $X_2 \subset X$ be nonempty and $f_0 : X_2 \to R^1$. f_0 is said to be level-bounded on X_2 if, for each $t \in R^1$, the set $\{x \in X_2 : f_0(x) \leq t\}$ is bounded.

Proposition 2.3 *Assume that* X *is a finite dimensional space,* $Y = R^l$, $C = R^l_+$. *Further assume that one of the following conditions holds:*

(1) *for each* $i \in \{1, \ldots, l\}$, f_i *is level-bounded on* X_1;
(2) *there exists* $\delta > 0$ *such that for each* $i \in \{1, \ldots, l\}$, f_i *is level-bounded on* $X_1(\delta)$, *where* $X_1(\delta)$ *is defined by (46); and*
(3) *for each* $i \in \{1, \ldots, l\}$,

$$\lim_{x\in X_1, \|x\|\to+\infty} max\{f_i(x), d_{X_0}(x)\} = +\infty. \tag{53}$$

Then (VP) is type III LP well-posed.

Proof Clearly, (1)⇒(3)⇒(2). So we need only to prove that if (2) holds, then (VP) is type III LP well-posed. Let $\{x_k\}$ be a type III LP minimizing sequence. Then (48) and (49) hold. (49) implies that $x_k \in X_1(\delta), \forall k \geq k_0$ for some $k_0 > 0$. (48) implies that there exists $0 < \epsilon_k \to 0$ such that

$$\xi(v - f(x_k)) \geq -\epsilon_k, \quad \forall v \in V. \tag{54}$$

We assert that $\{x_k\}$ is bounded. Otherwise, assume without loss of generality that $\|x_k\| \to +\infty$. Then, by the level-boundedness of each f_i on $X_1(\delta)$, we have

$$\lim_{k\to+\infty} f_i(x_k) = +\infty.$$

It follows that (54) cannot hold. Thus, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and $x^* \in X_1$ such that $x_{k_j} \to x^*$. This together with (49) implies that $x^* \in X_0$. Now we show that $x^* \in X^*$. Otherwise, there exist $x_0 \in X_0$ and $\delta_0 > 0$ such that

$$f_i(x_0) \le f_i(x^*), \quad i = 1, \ldots, l. \tag{55}$$

It is obvious that the set

$$A = \{x \in X_0 : f_i(x) \le f_i(x_0), i = 1, \ldots, l\}$$

is nonempty and compact. Note that $x_0 \in A$. It follows that there exists $\bar{x} \in A$ such that $f(x) - f(\bar{x}) \notin -C\backslash\{0\}, \forall x \in A$. It is easily verified that $\bar{x} \in X^*$. Moreover, by $\bar{x} \in A$, we have

$$f(\bar{x}) \le_C f(x_0).$$

This together with (55) implies that

$$f(\bar{x}) \le_C f(x^*) - \delta_0 e.$$

From $x_{k_j} \to x^*$ and the continuity of f on X_1, we have

$$f(\bar{x}) \le_C f(x_{k_j}) - \delta/2e,$$

when j is large enough, contradicting (54). The proof is complete. □

Similarly, we can prove the next result.

Proposition 2.4 *Assume that X is a finite dimensional space, $Y = R^l$, $C = R^l_+$. Further assume that one of the following conditions holds*:

(1) for each $i \in \{1, \ldots, l\}$, f_i is level-bounded on X_1;
(2) there exists $\delta > 0$ such that for each $i \in \{1, \ldots, l\}$, f_i is level-bounded on $X_2(\delta)$, where $X_2(\delta)$ is defined by (47); and
(3) for each $i \in \{1, \ldots, l\}$,

$$\lim_{x\in X_1, \|x\|\to+\infty} \max\{f_i(x), d_K(g(x))\} = +\infty. \tag{56}$$

Then (VP) is type III LP well-posed in the generalized sense.

Now, we consider the case when Z is a normed space and K is a closed and convex cone with nonempty interior int$K$ and let $e' \in \text{int}K$. Let $t \ge 0$ and denote

$$X_3(t) = \{x \in X_1 : g(x) \in K - te'\}. \tag{57}$$

Proposition 2.5 *Let Z be a normed space and K a closed and convex cone with nonempty interior* int$K$ *and let $e' \in \text{int}K$. If there exists $t_0 > 0$ such that $X_3(t_0)$ is compact, then (VP) is type III LP well-posed in the generalized sense.*

Proof According to (2) of Proposition 2.2, we need only to show that there exists $\delta_0 > 0$ such that $X_2(\delta_0)$ is compact. To this purpose, we need only to show that there exist $\delta_0 > 0$ such that $X_2(\delta_0) \subset X_3(t_0)$. Suppose to the contrary that there exists $0 < \delta_k \to 0$ and $x_k \in X_2(\delta_k)$ such that $x_k \notin X_3(t_0)$. That is,

$$d_K(g(x_k)) \le \delta_k, \tag{58}$$

$$g(x_k) \notin K - t_0 e'. \tag{59}$$

Define

$$\eta(z) = \min\{t \in R^1 : z \in -K + te'\}, \quad \forall z \in Z.$$

It is obvious that the function η has the same properties as the function ξ. From (59), we get

$$\eta(-g(x_k)) \ge t_0, \quad \forall k. \tag{60}$$

From (58), we deduce that there exists $w_k \in K$ such that $\|g(x_k) - w_k\| \to 0$. Let $z_k = w_k - g(x_k) \to 0$. Then, $-g(x_k) = z_k - w_k$, implying $\eta(-g(x_k)) \le \eta(z_k) \to 0$, contradicting (60). The proof is complete. □

Proposition 2.6 *Assume that X is a finite dimensional space,* $Y = R^l$, $C = R^l_+$, $e = (1, \ldots, 1) \in R^l$. *Let Z be a normed space and K a closed and convex cone with nonempty interior* int$K$ *and let* $e' \in$ int$K$. *Further assume that one of the following conditions holds:*

(1) for each $i \in \{1, \ldots, l\}$, f_i is level-bounded on X_1;
(2) there exists $t_0 > 0$ such that for each $i \in \{1, \ldots, l\}$, f_i is level-bounded on $X_3(t_0)$; and
(3) for each $i \in \{1, \ldots, l\}$, (56) holds. Then, (VP) is type III LP well-posed in the generalized sense.

Proof It is easy to show that (1)⇒(3)⇒(2). Similar to proof of Proposition 2.5, we can show that (2) implies that there exists $\delta_0 > 0$ such that for each $i \in \{1, \ldots, l\}$, f_i is level-bounded on $X_2(\delta_0)$. By (2) of Proposition 2.4 (VP) is type III LP well-posed in the generalized sense. □

Now we make the following assumption.

Assumption 2.1 X is a finite dimensional normed space, $Y = R^l, C = R^l_+$ $X_1 \subset X$ is a nonempty, closed and convex set, $K \subset Y$ is a closed and convex cone with nonempty interior int$K$ and $e' \in$ int$K$, each $f_i (i = 1, \ldots, l)$ is a convex function on X_1 and g is K-concave on X_1 (namely, for any $x_1, x_2 \in X_1$ and any $\theta \in (0, 1)$, there holds that $g(\theta x_1 + (1 - \theta)x_2) - \theta g(x_1) - (1 - \theta)g(x_2) \in K$).

It is obvious that under Assumption 2.1 (VP) is a convex vector program.

The next result was obtained in ([9], Theorem 2.1).

Lemma 2.1 *Let Assumption* 2.1 *hold. Then the following statements are equivalent:*

(1) *the optimal set* X^* *of (VP) is nonempty and compact;*
(2) *for each* $i \in \{1, \ldots, l\}$, *for any* $t \ge 0$, f_i *is level-bounded on the set* $X_3(t)$ *defined by (57).*

Theorem 2.9 *Let Assumption 2.1 hold. Then* (VP) *is type* III *LP well-posed in the generalized sense if and only if the optimal set* X^* *of* (VP) *is nonempty and compact.*

Proof The sufficiency part follows directly from Lemma 2.1 and Proposition 2.6, while the necessity part is obvious by (3) of Remark 1. □

Lemma 2.2 *Let Assumption* 2.1 *hold. Then the following statements are equivalent:*

(1) the optimal set X^* *of (VP) is nonempty and compact;*
(2) for each $i \in \{1, \cdots, l\}$*, for any* $\delta \geq 0$*,* f_i *is level-bounded on the set* $X_1(\delta)$ *defined by (46).*

Proof It is clear that problem (VP) is equivalent to the following vector optimization problem

$$\text{(VP')} \quad \begin{aligned} &\inf f(x) \\ &\text{s.t. } d_{X_0}(x) \leq 0. \end{aligned}$$

By Assumption 2.1, X_0 is nonempty and convex. It follows that $d_{X_0}(\cdot)$ is a continuous and convex function. Applying Lemma 2.1 by setting $g(x) = d_{X_0}(x), \forall x \in X_1, Z = R^1$ and $K = R^1_+$, we see that X^* is nonempty and compact if and only if each f_i is level-bounded on $X_1(\delta), \forall \delta \geq 0, i \in \{1, \ldots, l\}$. □

The following theorem follows immediately from (2) of Proposition 2.3 and Lemma 2.2.

Theorem 2.10 *Let Assumption* 2.1 *hold. Then* (VP) *is type* III *LP well-posed if and only if the optimal set* X^* *of (VP) is nonempty and compact.*

Remark 2 By Theorems 2.9 and 2.10 as well as (1) of Remark 1.1, if Assumption 2.1 holds, then any type of (generalized) LP well-posednesses is equivalent to the fact that the set X^* is nonempty and compact.

3 Relations among various types of (generalized) LP well-posedness

Simple relationships among the (generalized) LP well-posednesses were mentioned in (2) of Remark 1. Under Assumption 2.1, the equivalence of all the six types of (generalized) LP well-posednesses was noted in Remark 2. In this section, we investigate further relationships among them.

Theorem 3.1 *Suppose that there exist* $\delta > 0$*,* $\alpha > 0$*, and* $c > 0$*, such that*

$$d_{X_0}(x) \leq c d_K^{\alpha}(g(x)), \quad \forall x \in X_2(\delta), \tag{61}$$

where $X_2(\delta)$ *is defined by* (47)*. If (VP) is type I (resp. types* II *and* III*) LP well-posed, then (VP) is type I (resp. types* II *and* III*) LP well-posed in the generalized sense.*

Proof The proof is elementary. □

Remark 3 (61) is an error bound condition for the set X_0 in terms of the residual function

$$r(x) = d_K(g(x)), \quad \forall x \in X_1.$$

It is worth mentioning that this error bound condition has been intensively and extensively studied (see, e.g. [2, 4, 15] and the references therein).

Definition 3.3 **(1)** Let W be a topological space and $F: W \to 2^X$ be a set-valued map. F is said to be upper Hausdorff semicontinuous (u.H.c. in short) at $w \in W$ if, for any $\epsilon > 0$, there exists a neighbourhood U of w such that $F(U) \subset B(F(w), \epsilon)$, where for $Z \subset X$ and $r > 0$

$$B(Z, r) = \{x \in X : d_Z(x) \leq r\}.$$

It is clear that $X_2(\delta)$ given by (47) can be seen as a set-valued map from R^1_+ to X. Thus, we have the following theorem.

Theorem 3.2 *Assume that the set-valued map $X_2(\delta)$ defined by* (47) *is u.H.c. at* $0 \in R^1_+$. *If (VP) is type I (resp. types* II *and* III*) LP well-posed, then (VP) is type* I *(resp. types* II*, and* III*) LP well-posed in the generalized sense.*

Proof We prove only the type I case, the other two cases can be similarly proved. Let $\{x_k\} \subset X_1$ be a type I generalized LP minimizing sequence. That is,

$$d_V(f(x_k)) \to 0, \tag{62}$$

$$d_K(g(x_k)) \to 0. \tag{63}$$

(63), together with the u.H.c. of $X_2(\delta)$ at 0, implies that $d_{X_0}(x_k) \to 0$. This fact combined with (62) implies that $\{x_k\}$ is a type I LP minimizing sequence. Thus, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $x^* \in X^*$ such that $x_{k_j} \to x^*$. Hence, (VP) is type I LP well-posed in the generalized sense. □

Now, we consider the case when Z is a normed space.

Lemma 3.1 *Let Z be a normed space and $\{x_k\} \subset X_1$. Then, $d_K(g(x_k)) \to 0$ if and only if there exists $\{z_k\} \subset Z$ with $z_k \to 0$ such that $g(x_k) \in K + z_k, \forall k$.*

Proof Necessity. From $d_K(g(x_k)) \to 0$, we deduce that there exists $\{u_k\} \subset K$ such that

$$\|g(x_k) - u_k\| \to 0.$$

Let $z_k = g(x_k) - u_k$. Then, $z_k \to 0$ and $g(x_k) \in K + z_k$.

Sufficiency. Since $g(x_k) - z_k \in K$,

$$d_K(g(x_k)) \leq \|g(x_k) - (g(x_k) - z_k)\| = \|z_k\| \to 0.$$ □

Let

$$X_4(z) = \{x \in X_1 : g(x) \in K + z\}, \quad \forall z \in Z. \tag{64}$$

Clearly, $X_4(z)$ can seen as a set-valued map from Z to X.

Corresponding to Theorem 3.2, we have the following result.

Theorem 3.3 *Assume that the set-valued map $X_4(z)$ defined by* (64) *is u.H.c. at* $0 \in Z$. *If (VP) is type I (resp. types* II*, and* III*) LP well-posed, then (VP) is type I (resp. types II, and III) LP well-posed in the generalized sense.*

In the special case when K is a closed and convex cone with nonempty interior int$K$ and $e' \in \text{int}K$. We consider $X_3(t)$ defined by (57) as a set-valued map from R^1_+ to X. We have the next result.

Theorem 3.4 *Assume that the set-valued map $X_3(t)$ defined by (57) is u.H.c. at $0 \in R^1_+$. If (VP) is type I (resp. types II, and III) LP well-posed, then (VP) is type I (resp. types II and III) LP well-posed in the generalized sense.*

To end this section, we present the following theorem.

Theorem 3.5 *Assume that there exists $\delta_0 > 0$ such that g is uniformly continuous on the set $X_1(\delta_0)$ defined by (46). If (VP) is type I (resp. types II, and III) LP well-posed in the generalized sense, then (VP) is type I (resp. types II, and III) LP well-posed.*

Proof We prove only the type I case. Suppose that $\{x_k\} \subset X_1$ is a type I LP minimizing sequence. That is,

$$d_V(f(x_k)) \to 0, \tag{65}$$

$$d_{X_0}(x_k) \to 0. \tag{66}$$

By (66), we have $d_{X_0}(x_k) \leq \delta_0$ when $k \geq k_0$ for some $k_0 > 0$. By the uniform continuity of g on $X_1(\delta_0)$, $d_K(g(x_k)) \to 0$. This together with (65) implies that $\{x_k\}$ is a type I generalized LP minimizing sequence. Thus, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $x^* \in X^*$ such that $x_{k_j} \to x^*$. Hence (VP) is type I LP well-posed. □

4 Application to a class of penalty methods

In this section, we consider the convergence of a class of penalty methods under the assumption of type III generalized LP well-posedness of (VP).

Let $\alpha > 0$ and $e \in \text{int}C$. Consider the following penalty problem for (VP):

$$(VPP_\alpha(r)) \quad \inf_{x \in X_1} f(x) + r d^\alpha_K(g(x))e, \quad r > 0.$$

Remark 4 This class of penalty methods was studied in, e.g. [9].

Theorem 4.1 *Let $0 < r_n \to +\infty$. Consider problems (VP) and* $(\text{VPP}_\alpha(r_k))$. *Assume that there exist $\bar{r} > 0$ and $m_0 \in R^1$ such that*

$$f(x) + \bar{r} d^\alpha_K(g(x))e \geq_C m_0 e, \quad \forall x \in X_1. \tag{67}$$

Let $0 < \epsilon_k \to 0$. Suppose that each $x_k \in X_1$ satisfies

$$f(x) + r_k d^\alpha_K(g(x))e - f(x_k) - r_k d^\alpha_K(g(x_k))e + \epsilon_k e \notin -\text{int}C, \quad \forall x \in X_1. \tag{68}$$

Further assume that (VP) is type III LP well-posed in the generalized sense. Then, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $x^* \in X^*$ such that $x_{k_j} \to x^*$. Moreover, each limit point of $\{x_k\}$ belongs to X^*.

Proof Let $x_0 \in X_0$. From (68), we deduce that

$$f(x_0) - f(x_k) - r_k d_K^{\alpha}(g(x_k))e + \epsilon_k e \notin -\text{int}C. \tag{69}$$

The combination of (67) and (69) yields

$$f(x_0) - m_0 e - (r_k - \bar{r}) d_K^{\alpha}(g(x_k))e + \epsilon_k e \notin -\text{intC}$$

implying

$$\xi(f(x_0)) - m_0 - (r_k - \bar{r}) d_K^{\alpha}(g(x_k)) + \epsilon_k \geq 0$$

namely,

$$d_K(g(x_k)) \leq \left[\frac{\xi(f(x_0)) + \epsilon_k - m_0}{r_k - \bar{r}}\right]^{1/\alpha}.$$

Hence,

$$\lim_{k \to +\infty} d_K(g(x_k)) = 0. \tag{70}$$

Moreover, from (69), we have

$$f(x_0) - f(x_k) + \epsilon_k e \notin -\text{int}C.$$

By the arbitrariness of $x_0 \in X_0$, this further implies that

$$v - f(x_k) + \epsilon_k e \notin -\text{int}C, \quad \forall v \in V.$$

Therefore,

$$\xi(v - f(x_k)) + \epsilon_k \geq 0, \quad \forall v \in V.$$

Hence,

$$\liminf_{k \to +\infty} \left\{ \inf_{v \in V} \xi(v - f(x_k)) \right\} \geq 0. \tag{71}$$

By (70) and (71), $\{x_k\}$ is a type III generalized LP minimizing sequence. Since (VP) is type III LP well-posed in the generalized sense, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $x^* \in X^*$ such that $x_{k_j} \to x^*$. Finally, suppose that $\bar{x}$ is a limit point of $\{x_k\}$. Then, there exists a subsequence $\{x_{k_j}\}$ such that $x_{k_j} \to \bar{x}$. It is obvious that $\{x_{k_j}\}$ is also a type III generalized LP minimizing sequence. By the type III generalized LP well-posedness of (VP), there exist a subsequence $\{x_{k_{j_l}}\}$ and some $\bar{x}' \in X^*$ such that $x_{k_{j_l}} \to \bar{x}'$. On the other hand, we have $x_{k_{j_l}} \to \bar{x}$. It follows that $\bar{x} = \bar{x}'$. Hence, $\bar{x} \in X^*$. The proof is complete. □

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References

1. Bednarczuk, E., Penot, J.P.: Metrically well-set minimization problems. Appl. Math. Optim. **26**, 273–285 (1992)
2. Bosch, P., Jourani, A., Henrion, R.: Sufficient conditions for error bounds and applications. Appl. Math. Optim. **50**, 161–181 (2004)

3. Deng, S.: Coercivity properties and well-posedness in vector optimization. RAIRO Oper. Res. **37**, 195–208 (2003)
4. Dontchev, A.L., Rockafellar, R.T.: Regularity properties and conditioning in variational analysis and optimization. Set-Valued Analysis, **12**, 79–109 (2004)
5. Dontchev, A.L., Zolezzi, T.: Well-Posed Optimization Problems, Lecture Notes in Mathematics, 1543. Springer, Berlin (1993)
6. Furi, M., Vignoli, A.: About well-posed minimization problems for functionals in metric spaces. J. Optim. Theory Appl. **5**, 225–229 (1970)
7. Huang, X.X.: Extended well-posed properties of vector optimization problems. J. Optim. Theory Appl. **106**, 165–182 (2000)
8. Huang, X.X., Yang, X.Q.: Duality and exact penalization for vector optimization via augmented Lagrangian. J. Optim. Theory Appl. **111**, 615–640 (2001)
9. Huang, X.X., Yang, X.Q., Teo, K. L.: Characterzing the nonemptiness and compactness of solution set of a convex optimization problem with cone constraints and applications. J. Optim. Theory Appl. 123 391–407 (2004)
10. Konsulova, A.S., Revalski, J.P.: Constrained convex optimization problems-well-posedness and stability. Num. Funct. Anal. Optim. **15**, 889–907 (1994)
11. Kuratowski, C.: Topologie, vol. 1. Panstwowe Wydawnicto Naukowa, Warszawa, Poland (1958)
12. Levitin, E.S, Polyak, B. T.: Convergence of minimizing sequences in conditional extremum problems. Soviet Math. Dokl. **7**, 764–767 (1966)
13. Luc, D. T.: Theory of Vector Optimization. Springer-Verlag, Berlin (1989)
14. Lucchetti, R., Revalski, J.: (eds.): Recent Developments in Well-Posed Variational Problems. Kluwer Academic Publishers, Dordrecht (1995)
15. Pang, J. S.: Error bounds in mathematical programming. Math. Program. **79**, 299–332 (1997)
16. Tykhonov, A. N.: On the stability of the functional optimization problem. USSR Compt. Math. Math. Phys. **6** 28–33 (1966)
17. Zolezzi, T.: Extended well-posedness of optimization problems. J. Optim. Theory Appl. **91**, 257–266 (1996)